STUDY OF NUMERICAL SOLUTIONS OF VARIABLE – ORDER FRACTIONAL DIFFERENTIAL EQUATIONS USING DIFFERENT METHODS

A proposed synopsis submitted to

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Introduction

Fractional calculus is a branch of mathematics that extends traditional concepts of differentiation and integration to non-integer (fractional) orders. Similar to how fractional exponents generalize integer powers, fractional derivatives provide a powerful tool for modeling systems with memory and hereditary effects. The origins of this field date back to 1695 when L'Hôpital inquired about the meaning of $\frac{d^n y^m}{dx^n}$ if $m = 1, n = 1/2$. Leibniz responded by suggesting that $\frac{d^{1/2}}{dx^{1/2}}x$ will be equal to $2\frac{\sqrt{x}}{\sqrt{\pi}}$. Over time, the concept evolved into a formal mathematical theory, especially during the 19^{th} century (Ross, 1977) [1].

Fractional differential equations have emerged as a more precise means of modeling physical systems compared to conventional differential equations. The advantages of employing fractional calculus in various research areas have been extensively established[2, 3 , 4]. Numerous dynamical systems are modeled and controlled using fractional calculus, leveraging its unique advantages. However, the challenges of obtaining exact solutions for fractional-orde differential equations have led to a growing focus on the development of numerical techniques for solving them. Techniques such as the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Differential Transform Method (DTM), and Variational Iteration Method (VIM) have been proposed. These methods have significantly broadened the applicability of fractional calculus in fields such as rheology, biology, control systems, and materials science.

Fractional calculus deals with integrals and derivatives of functions, but its focus is on non-integer order integrals and derivatives, as opposed to the integer order counterparts. These fractional integrals and derivatives can be of real or complex orders, encompassing integer orders as a subset. Notably, fractional derivatives introduce a crucial memory mechanism to systems, making them indispensable in various contexts. Therefore, understanding the benefits of incorporating fractional derivatives into classical equations holds significance.

Some useful definitions

This research utilizes several definitions of fractional derivatives:

1. Riemann-Liouville fractional integral [5]

$$ {}_{a+}I_t^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)\, ds. $$

2. Riemann-Liouville fractional derivative [5]

$$\begin{aligned} {}^{RL}_{a+}D^{\alpha}_{t}y(t) &= D^{n}I^{n-\alpha}_{t}y(t), \quad D^{n} = \frac{d^{n}}{dt^{n}} \\ &= \frac{d^{n}}{dt^{n}}\left[\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}y(s)\,ds\right] \end{aligned}$$

3. Caputo fractional derivative [5]

$$\begin{aligned} {}^{RL}_{a+}D^{\alpha}_{t}y(t) &= I^{n-\alpha}_{t}D^{n}y(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}y(s)^{(n)}\,ds \end{aligned}$$

4. Hilfer fractional derivative [6]

$${}^{H}_{a+}D^{\alpha,\beta}_{t}y(t) = I^{\beta(n-\alpha)}_{a+}\left(\frac{d}{dt}\right)^{n} I^{(1-\beta)(n-\alpha)}_{a+}y(t)$$

5. Generalized Riemann-Liouville fractional integral [5, 7]

$${}_{a+}I^{\alpha,\psi}_{t}y(t) = \frac{1}{\Gamma(\alpha)}\int_{a}^{t}\psi'(\xi)\left(\psi(t)-\psi(\xi)\right)^{\alpha-1}y(\xi)\,d\xi$$

6. Generalized Riemann-Liouville fractional derivative [5, 7]

$${}^{RL}_{a+}D^{\alpha,\psi}_{t}y(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}{}_{a+}I^{n-\alpha,\psi}_{t}y(t)$$

7. Generalized Caputo fractional derivative [8]

$${}^{C}_{a+}D^{\alpha,\psi}_{t}y(t) = {}_{a+}I^{n-\alpha,\psi}_{t}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}y(t)$$

8. Generalized Hilfer fractional derivative [9]

$${}^{H}_{a+}D^{\alpha,\beta,\psi}_{t}y(t) = {}_{a+}I^{\beta(n-\alpha)\psi}_{t}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}a{+}I^{(1-\beta)(n-\alpha)\psi}_{t}y(t)$$

9. Caputo-Fabrizio fractional integral [10]

$${}^{CF}_{a+}I^{\alpha}_{t}y(t) = \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}y(t) + \frac{2\alpha}{(2-\alpha)N(\alpha)}\int_{a}^{t}y(s)\,ds$$

10. Caputo-Fabrizio fractional derivative [10]

$$ {}^{CF}_{a+}D^{\alpha}_{t}y(t) = \frac{(2-\alpha)N(\alpha)}{2(1-\alpha)}\int_{a}^{t}\exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right)y'(s)\,ds $$

11. Atangana-Baleanu fractional integral [11]

$$ {}^{AB}_{a+}I^{\alpha}_{t}y(t) = \frac{1-\alpha}{N(\alpha)}y(t) + \frac{\alpha}{N(\alpha)\Gamma(\alpha)}\int_{a}^{t}(t-s)^{\alpha-1}y(s)\,ds $$

12. Atangana-Baleanu-Caputo fractional derivative [11]

$$ {}^{ABC}_{a+}D^{\alpha}_{t}y(t) = \frac{N(\alpha)}{1-\alpha}\int_{0}^{t}E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right)y'(s)\,ds $$

13. Tempered-Caputo fractional derivative [12]

$$ {}^{TC}_{a+}D^{\alpha}_{t}y(t) = (e^{-\lambda t})^{C}_{a+}D^{\alpha,\psi}_{t}\left(e^{\lambda t}y(t)\right) $$

14. Conformable fractional derivatives [13, 14]

$$ T_{\alpha}y(t) = \lim_{\epsilon\to 0}\frac{y(t+\epsilon t^{1-\alpha})-y(t)}{\epsilon}, \quad 0<\alpha<1 $$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number $\alpha, 0 \le \beta \le 1, \psi$ be an increasing function such that $\psi'(t) \neq 0, \mathcal{N}(\alpha)$ is the normalization function which is defined as $\mathcal{N}(0) = \mathcal{N}(1) = 1$, and $\mathbb{E}_{\alpha}$ is called the Mittag-Leffler function defined by the series

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^{K}}{\Gamma(k\alpha+1)}, $$

Review of Literature

Fractional calculus has matured into a robust modeling framework, capable of capturing the memory-dependent and nonlocal characteristics of complex systems. Unlike integer-order derivatives, fractional derivatives account for past states, making them ideal for time-varying and non-stationary systems. To further enhance flexibility, the concept of variable-order (VO) fractional operators has been introduced, wherein the order of differentiation varies with respect to time, space, or other parameters [15].

Samko and Ross (1993) first introduced the concept of VO integrals and derivatives [16], and Lorenzo and Hartley later unified various definitions [17], laying the groundwork for further advancements. VO-FDEs have since been extensively studied and applied in areas

such as anomalous diffusion, viscoelasticity, control systems, and petroleum engineering [18, 19, 20 21].

Applications of VO-FDEs include modeling fractal dynamics (Kobelev et al.) [22], viscoelastic oscillators (Coimbra et al.) [23], and epidemiological systems (Sweilam and Al-Mekhlafi) [24]. Liu et al. investigated stability and convergence of finite-difference schemes [25], while Zayernouri and Karniadakis proposed exponentially accurate spectral collocation methods [26]. Further contributions include Chen et al.'s implicit schemes for 2D percolation equations and Zhao et al.'s second-order approximation for time-dependent fractional derivatives [27]. Zhao et al. devised second-order approximation formulae for time-dependent VO fractional derivatives [28], enabling the description of anomalous diffusion and wave propagation.Cao and Qiu introduced a high-order numerical method for VO-FDEs [29], and other works have explored the analytical inversion and structure of variable-order operators using Riemann–Liouville formulations and Fourier transforms.

Despite these developments, challenges persist in the numerical treatment of VO-FDEs, particularly regarding stability, computational cost, and benchmarking.

Recent Advances and Innovations

In recent years, the study of numerical solutions for VO-FDEs has witnessed remarkable advancements and innovations. Researchers from diverse disciplines have explored new frontiers in the realm of fractional calculus, focusing on addressing complex challenges posed by systems with changing fractional orders. Some of the notable recent advances include:

- **Adaptive Numerical Schemes:** Researchers have been developing adaptive numerical schemes that adjust their parameters according to the variable fractional orders. These schemes offer enhanced accuracy and efficiency by dynamically tailoring their resolution to capture abrupt changes in fractional orders.

- **Hybrid Algorithms:** A fusion of different numerical techniques, such as finite difference methods, spectral methods, and neural networks, has emerged as an effective approach to tackle the complexities of VO-FDEs. These hybrid algorithms leverage the strengths of individual methods to provide accurate solutions across a wide range of scenarios.

- **Fractional Time-Stepping Techniques:** Innovative time-stepping techniques designed specifically for VO-FDEs have been devised. These methods account for the changing fractional orders at each time step, enabling more precise simulations of dynamic processes with evolving memory effects.

- **Parallel Computing Strategies:** With the growing computational demands solving VO-FDEs, parallel computing strategies have gained traction. Utilizing the power of modern parallel architectures, researchers are able to accelerate the computation of solutions for large-scale and real-time applications.

These advancements have significantly improved the precision, adaptability, and efficiency of numerical solutions for VO-FDEs.

Applications and Challenges

The study of numerical solutions for VO-FDEs holds immense potential across various scientific and engineering domains. Recent applications have demonstrated the versatility and significance of this research area. Some notable applications include:

- **Biomedical Engineering:** In the field of biomedical engineering, VO-FDEs are utilized to model complex physiological processes that exhibit variable memory characteristics. These models aid in understanding and predicting the behavior of biological systems under changing conditions.

- **Geophysics and Earth Sciences** Earth processes, such as seismic wave propagation and sediment transport, often exhibit non-integer behaviors. The use of VO-FDEs helps capture the intricate memory effects present in these systems, leading to more accurate seismic hazard assessments and geological predictions.

- **Materials Science:** Understanding the behavior of viscoelastic materials with changing properties is crucial in materials science. VO-FDEs provide a robust framework to simulate and analyze the time-dependent mechanical response of materials, contributing to the design of advanced materials with tailored properties.

- **Control Systems:** Dynamic systems subject to variable order fractional dynamics present challenges for control engineers. The study of numerical solutions for VO-FDEs enables the design of controllers that account for changing memory effects, enhancing the stability and performance of control systems.

Despite these promising applications, challenges persist in the study of numerical solutions for VO-FDEs:

- **Algorithm Stability:** Adapting numerical methods to changing fractional orders while ensuring stability remains a challenge. Ensuring the stability of algorithms under varying conditions is crucial for accurate and reliable results.

- **Computational Efficiency:** The computational demands of solving VO-FDEs can be substantial, particularly for high-dimensional systems. Developing efficient algorithms that balance accuracy and computational cost is a continuing endeavor.
- **Numerical Accuracy:** Achieving high accuracy in numerical solutions for VO-FDEs requires careful consideration of discretization methods and convergence criteria. Striking a balance between accuracy and computational efficiency is an ongoing concern.
- **Validation and Benchmarking:** Validating numerical solutions against analytical solutions or experimental data is essential to establish the reliability and effectiveness of the proposed methods. Benchmarking against established problems aids in assessing the performance of new algorithms.

In this proposed study, we aim to delve into these recent advances, applications, and challenges in the field of numerical solutions for variable order fractional differential equations. By exploring innovative numerical techniques, applying them to diverse applications, and addressing existing challenges, we strive to contribute to the advancement of this dynamic and impactful research area.

Objectives of the Proposed Research

The primary objectives of this study are as follows:

1. To explore the mathematical foundations of variable order fractional differential equations and their significance in modeling real-world phenomena.
2. To review and compare various numerical methods commonly employed for solving fractional differential equations, including finite difference methods, fractional Adams-Bashforth-Moulton methods, and spectral methods.
3. To develop and implement numerical algorithms for solving VOFDEs with varying degrees of complexity.
4. To analyze the accuracy, stability, and convergence properties of the implemented numerical methods.
5. To investigate the impact of variable fractional orders on the behavior of solutions to differential equations and compare them with constant order cases.
6. To apply the developed methods to practical problems from different scientific and engineering domains to showcase their applicability and effectiveness.

Research Methodology

The study will be conducted in the following stages:

1. **Literature Review:** A comprehensive review of the existing literature on fractional calculus, variable order fractional differential equations, and numerical methods for their solutions will be performed.
2. **Method Selection and Implementation:** Various numerical methods suitable for solving VOFDEs will be selected and implemented. This includes finite difference methods, time-stepping algorithms, and spectral methods.
3. **Algorithm Analysis:** The implemented algorithms will be rigorously tested and analyzed in terms of accuracy, stability, and convergence. The methods' performance will be compared for a range of VOFDEs with different orders and complexities.
4. **Numerical Experiments:** Practical problems involving variable order fractional differential equations will be chosen from different fields such as physics, biology, and engineering. The developed numerical methods will be applied to these problems, and the results will be compared with available analytical solutions or other benchmark methods.
5. **Variable-Order Analysis:** The behavior of solutions to VOFDEs with variable fractional orders will be examined and compared to solutions with constant fractional orders. Insights into the impact of variable orders on system dynamics will be derived.

Expected Contributions

This study is expected to yield the following outcomes:

- A comprehensive analysis of numerical methods suitable for solving variable order fractional differential equations.
- Implemented numerical algorithms and codes that can be shared with the scientific community.
- Insights into the behavior of solutions with variable fractional orders compared to constant order cases.
- Recommendations for selecting appropriate numerical methods based on the nature of the problem.

Significance of the Study

This study contributes to the field of computational fractional calculus by providing efficient numerical techniques for solving VO-FDEs. These equations appear in a variety of real-world applications such as viscoelastic materials, diffusion processes, and electrical circuits. The development of stable and accurate algorithms will enhance our ability to simulate complex systems with variable memory.

Expected Outcomes

- Accurate and efficient numerical algorithms for VO-FDEs.
- Theoretical analysis of stability and convergence.
- Real-world case studies validating the proposed methods.
- Publication of results in peer-reviewed journals.

Plan of work:

Sr. No.	Period	Plan of work	Targets to be Achieved
1	First year	Collecting the papers and analyze them	Try to obtain new results based on the previous work
2	Second year	work on some numerical methods commonly used to solve VO-FDEs, including finite difference methods, spectral methods, and hybrid algorithms	Submit at least two papers
3	Third year	Publishing some papers and writing of the thesis.	Some more papers andthesis submission.

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